ECS315 2014/1 Part IV.2 Dr.Prapun

10.4 Families of Continuous Random Variables

Theorem 10.24 states that any nonnegative function $f(x)$ whose integral over the interval $(-\infty, +\infty)$ equals 1 can be regarded as a probability density function of a random variable. In real-world applications, however, special mathematical forms naturally show up. In this section, we introduce a couple families of continuous random variables that frequently appear in practical applications. The probability densities of the members of each family all have the same mathematical form but differ only in one or more parameters.

10.4.1 Uniform Distribution

Definition 10.35. For a uniform random variable on an interval $[a, b]$, we denote its family by uniform($[a, b]$) or $\mathcal{U}([a, b])$ or simply $\mathcal{U}(a, b)$. Expressions that are synonymous with "X is a uniform random variable" are "X is uniformly distributed" $^{\text{on } [a,b]}$, "X has a uniform distribution" $^{\text{on } [a,b]}$, and "X has a uniform density" $^{\text{on } [a,b]}$. This family is characterized by

$$f_X(x) = \begin{cases} 0, & x < a, x > b \\ \frac{1}{b-a}, & a \leq x \leq b \end{cases}$$

- The random variable X is just as likely to be near any value in $[a, b]$ as any other value.

$\sim \mathcal{U}(0,1)$

Ex. The number generated by the rand command in MATLAB is uniformly distributed on $[0,1]$

- In `MATLAB`,

  (a) use `X = a+(b-a)*rand` or `X = random('Uniform',a,b)` to generate the RV,

  (b) use `pdf('Uniform',x,a,b)` and `cdf('Uniform',x,a,b)` to calculate the pdf and cdf, respectively.

Exercise 10.36. Show that $F_X(x) = \begin{cases} 0, & x < a, \cancel{x > b} \\ \frac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$

$$\int_{-\infty}^{x} f_X(t)\,dt = P[X \le x] \equiv F_X(x)$$

$f_X(x)$

$\frac{1}{b-a}$

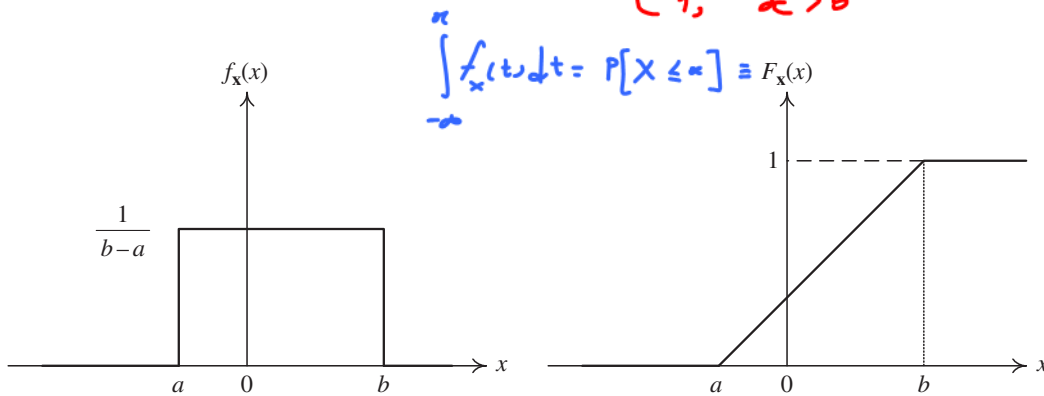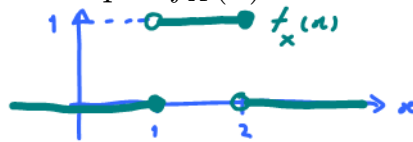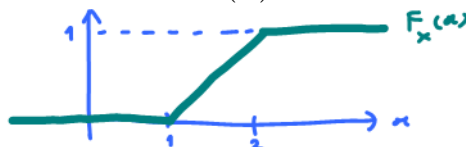a 0 b x

1

a 0 b x

Figure 16: The pdf and cdf for the uniform random variable. [16, Fig. 3.5]

Example 10.37 (F2011). Suppose X is uniformly distributed on the interval $(1,2)$. $(X \sim \mathcal{U}(1,2).)$

(a) Plot the pdf $f_X(x)$ of X.

$f_X(x)$
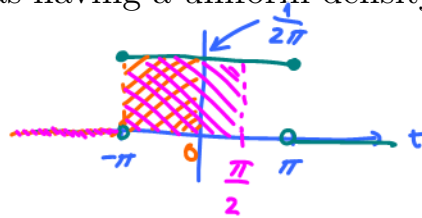
(b) Plot the cdf $F_X(x)$ of X.

$F_X(x)$

10.38. The uniform distribution provides a probability model for selecting a point at random from the interval $[a,b]$.

- Use with caution to model a quantity that is known to vary randomly between a and b but about which little else is known.

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Example 10.39. [9, Ex. 4.1 p. 140-141] In coherent radio communications, the phase difference between the transmitter and the receiver, denoted by Θ, is modeled as having a uniform density on $[-\pi, \pi]$.

(a) $P[\Theta \leq 0] = \frac{1}{2}$

$$= \pi \times \frac{1}{2\pi} = \frac{1}{2}$$

(b) $P\left[\Theta \leq \frac{\pi}{2}\right] = \frac{3}{4}$

$$\left(\underbrace{\frac{\pi}{2} - (-\pi)}_{\text{width}}\right) \times \underbrace{\frac{1}{2\pi}}_{\text{height}} = \frac{3\pi}{2} \times \frac{1}{2\pi} = \frac{3}{4}$$

$$\int_{-\infty}^{\infty} x \, \widetilde{f_X(x)} \, dx = \int_a^b x \, \frac{1}{b-a} \, dx = ?$$

Exercise 10.40. Show that when $X \sim \mathcal{U}([a,b])$, $\mathbb{E}X = \frac{a+b}{2}$, $\operatorname{Var} X = \frac{(b-a)^2}{12}$, and $\mathbb{E}\left[X^2\right] = \frac{1}{3}\left(b^2 + ab + a^2\right)$.

10.4.2 Gaussian Distribution

10.41. This is the ==most widely used model== for the distribution of a random variable. When you have many independent random variables, a fundamental result called the central limit theorem (CLT) (informally) says that the sum (technically, the average) of them can often be approximated by normal distribution.

Definition 10.42. *Gaussian* random variables:

(a) Often called ***normal*** random variables because they occur so frequently in practice.

(b) In `MATLAB`, use `X = random('Normal',m,`σ`)` or `X = `σ`*randn + m`.

(c) $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$.

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2} dt$$

- In `Excel`, use `NORMDIST(x,m,`σ`,FALSE)`.
  In `MATLAB`, use `normpdf(x,m,`σ`)` or `pdf('Normal',x,m,`σ`)`.
- Figure 17 displays the famous ***bell-shaped*** graph of the Gaussian pdf. This curve is also called the *normal* curve.

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(d) $F_X(x)$ has no closed-form expression. However, see 10.48.

- In MATLAB, use normcdf(x,m,σ) or cdf('Normal',x,m,σ).
- In Excel, use NORMDIST(x,m,σ,TRUE).

(e) We write $X \sim \mathcal{N}(m, \sigma^2)$.

$\mathcal{N}$

Var not std.

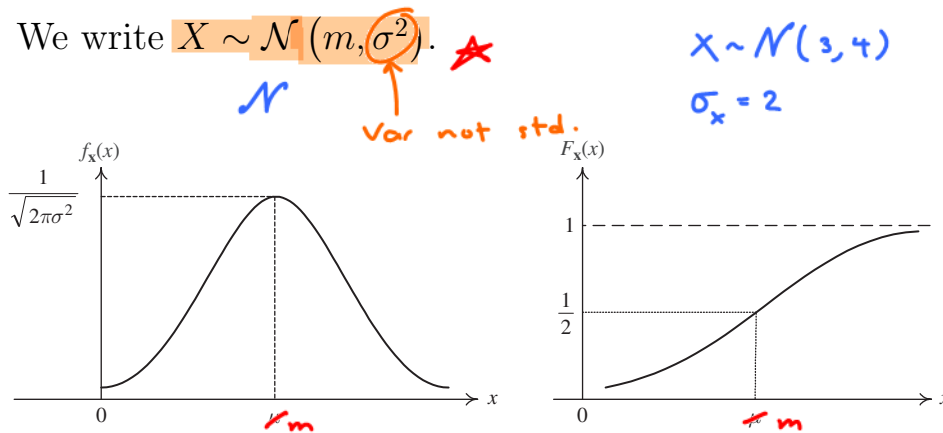$X \sim \mathcal{N}(3, 4)$

$\sigma_x = 2$

Figure 17: The pdf and cdf of $\mathcal{N}(\mu, \sigma^2)$. [16, Fig. 3.6]

10.43. $\mathbb{E}X = m$ and $\operatorname{Var} X = \sigma^2$.

10.44. Important probabilities:
$P[|X - \mu| < \sigma] = 0.6827$;
$P[|X - \mu| > \sigma] = 0.3173$;
$P[|X - \mu| > 2\sigma] = 0.0455$;
$P[|X - \mu| < 2\sigma] = 0.9545$
These values are illustrated in Figure 20.

Example 10.45. Figure 21 compares several deviation scores and the normal distribution:

(a) Standard scores have a mean of zero and a standard deviation of 1.0.

(b) Scholastic Aptitude Test scores have a mean of 500 and a standard deviation of 100.

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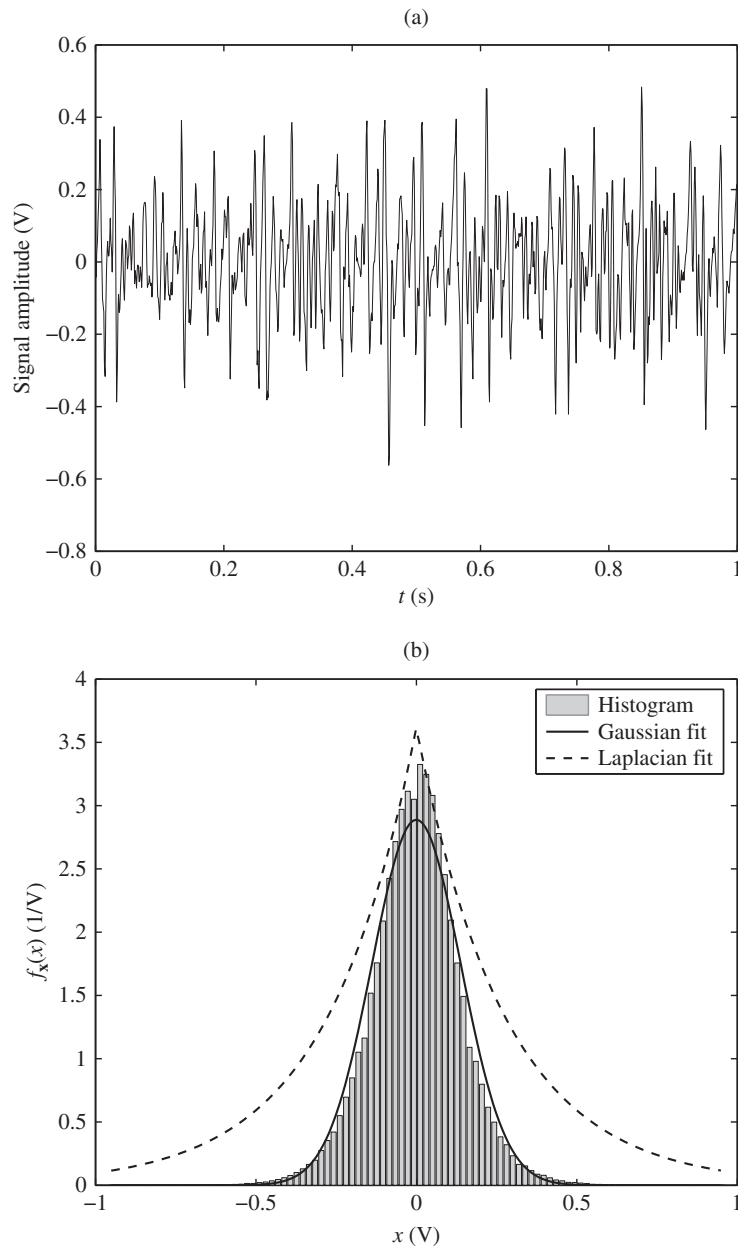
Figure 18: Electrical activity of a skeletal muscle: (a) A sample skeletal muscle (emg) signal, and (b) its histogram and pdf fits. [16, Fig. 3.14]
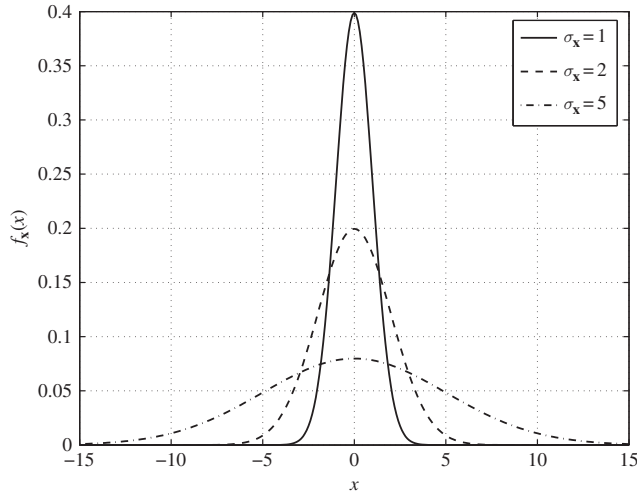
Figure 19: Plots of the zero-mean Gaussian pdf for different values of standard deviation, σ_X. [16, Fig. 3.15]
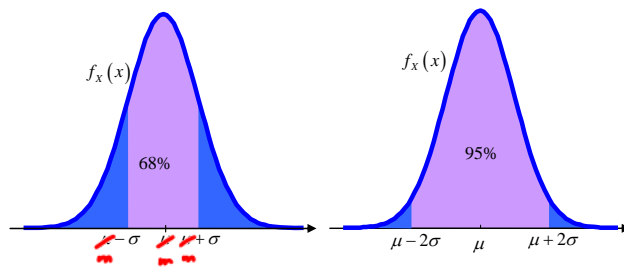

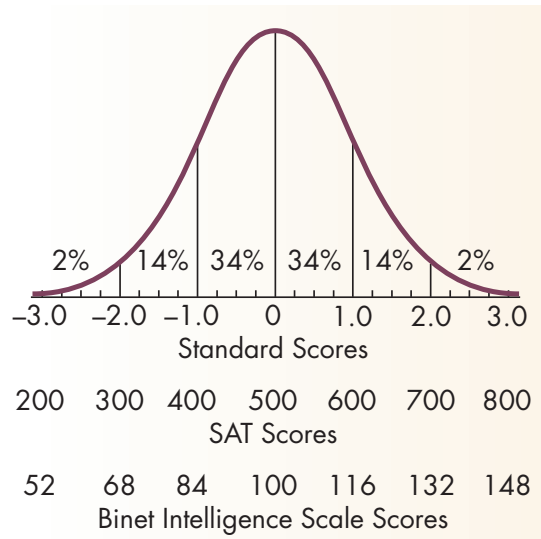Figure 20: Probability density function of $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

Figure 21: Comparison of Several Deviation Scores and the Normal Distribution

(c) Binet Intelligence Scale[43] scores have a mean of 100 and a standard deviation of 16.

In each case there are 34 percent of the scores between the mean and one standard deviation, 14 percent between one and two standard deviations, and 2 percent beyond two standard deviations. [Source: Beck, Applying Psychology: Critical and Creative Thinking.]
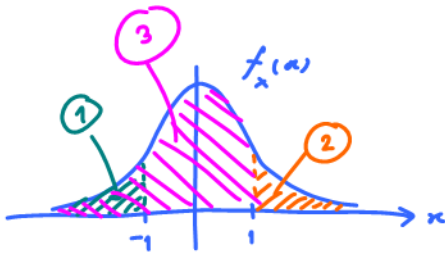
(handwritten: $m = 0$, $\sigma = 1$)

10.46. $\mathcal{N}(0,1)$ is the ***standard*** Gaussian (normal) distribution.

- In Excel, use NORMSINV(RAND()).
  In MATLAB, use randn.

  (handwritten: $f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$)

- The standard normal cdf is denoted by $\Phi(z)$.

  (handwritten: $= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$)

  ○ It inherits all properties of cdf.

  ○ Moreover, note that $\Phi(-z) = 1 - \Phi(z)$.

  (handwritten:
  $\Phi(-1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$

  $\bar{\Phi}(-1) = P[X \leq -1] = ① = ② = 1 - ③$

  $= 1 - P[X \leq 1] = 1 - \Phi(1)$
)

10.47. Relationship between $\mathcal{N}(0,1)$ and $\mathcal{N}(m, \sigma^2)$.

(a) An arbitrary Gaussian random variable with mean m and variance σ^2 can be represented as $\sigma Z + m$, where $Z \sim \mathcal{N}(0,1)$.

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[43]Alfred Binet, who devised the first general aptitude test at the beginning of the 20th century, defined intelligence as the ability to make adaptations. The general purpose of the test was to determine which children in Paris could benefit from school. Binets test, like its subsequent revisions, consists of a series of progressively more difficult tasks that children of different ages can successfully complete. A child who can solve problems typically solved by children at a particular age level is said to have that mental age. For example, if a child can successfully do the same tasks that an average 8-year-old can do, he or she is said to have a mental age of 8. The intelligence quotient, or IQ, is defined by the formula:

$$IQ = 100 \times (\text{Mental Age/Chronological Age})$$

There has been a great deal of controversy in recent years over what intelligence tests measure. Many of the test items depend on either language or other specific cultural experiences for correct answers. Nevertheless, such tests can rather effectively predict school success. If school requires language and the tests measure language ability at a particular point of time in a childs life, then the test is a better-than-chance predictor of school performance.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 3.1 The standard normal CDF $\Phi(y)$.

This relationship can be used to generate general Gaussian RV from standard Gaussian RV.

(b) If $X \sim \mathcal{N}\left(m, \sigma^2\right)$, the random variable

$$Z = \frac{X - m}{\sigma}$$

is a standard normal random variable. That is, $Z \sim \mathcal{N}(0,1)$.

- Creating a new random variable by this transformation is referred to as **standardizing**.

- The standardized variable is called "**standard score**" or "**z-score**".

10.48. It is impossible to express the integral of a Gaussian PDF between non-infinite limits (e.g., (20)) as a function that appears on most scientific calculators.

- An old but still popular technique to find integrals of the Gaussian PDF is to refer to tables that have been obtained by numerical integration.

  - One such table is the table that lists $\Phi(z)$ for many values of positive z.

  - For $X \sim \mathcal{N}\left(m, \sigma^2\right)$, we can show that the CDF of X can be calculated by

$$F_X(x) = \Phi\left(\frac{x - m}{\sigma}\right).$$

$$\mu = \frac{t-m}{\sigma} \qquad \frac{d\mu}{dt} = \frac{1}{\sigma} \qquad d\mu = \frac{1}{\sigma}dt$$
$$dt = \sigma\, d\mu$$

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-m}{\sigma}\right)^2} dt \overset{\underset{\mu = \frac{t-m}{\sigma}}{x=\frac{x-m}{\sigma}}}{=} \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2} d\mu = \Phi\left(\frac{x-m}{\sigma}\right)$$

standard gaussian

Example 10.49. Suppose $Z \sim \mathcal{N}(0,1)$. Evaluate the following probabilities.
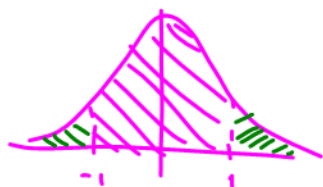
Recall: For cont. RV, $P\left[a \leq X \leq b\right] = F_X(b) - F_X(a)$

(a) $P\left[-1 \leq Z \leq 1\right]$

$$= F_Z(1) - F_Z(-1) = \Phi(1) - \Phi(-1)$$
$$= \Phi(1) - \left(1 - \Phi(1)\right)$$
$$= 2\,\Phi(1) - 1 = 0.6826$$

0.8413

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(b) $P[-2 \le Z \le 2]$ = $F_Z(2) - F_Z(-2)$ = $\bar{\Phi}(2) - \Phi(-2)$

$\underbrace{\quad}_{0.97725}$

= $\Phi(2) - (1 - \Phi(2))$ = $2\widetilde{\Phi(2)} - 1$ = 0.9545

Example 10.50. Suppose $X \sim \mathcal{N}(1, 2)$. Find $P[1 \le X \le 2]$.

$\hookrightarrow \sigma = \sqrt{2}$

= $\Phi(\frac{2-1}{\sqrt{2}}) - \Phi(\frac{1-1}{\sqrt{2}})$

= $\Phi(\frac{1}{\sqrt{2}}) - \Phi(0)$ = $0.7611 - 0.5$ = 0.2611

$\underbrace{\quad}_{0.7071}$

10.51. *Q-function*: $Q(z) = \int\limits_{z}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ corresponds to $P[X > z]$ where $X \sim \mathcal{N}(0, 1)$; that is $Q(z)$ is the probability of the "tail" of $\mathcal{N}(0, 1)$. The Q function is then a complementary cdf (ccdf).
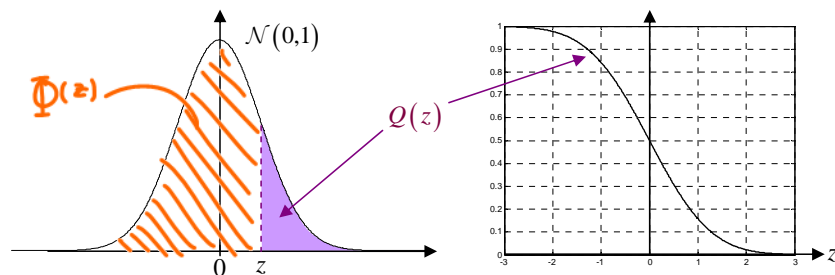
$\Phi(5) + Q(5) = 1$

Figure 22: Q-function

(a) Q is a decreasing function with $Q(0) = \frac{1}{2}$.

(b) $Q(-z) = 1 - Q(z) = \Phi(z)$

10.52. *Error function* (MATLAB): $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int\limits_{0}^{z} e^{-x^2} dx = 1 - 2Q(\sqrt{2}z)$

(a) It is an odd function of z.

(b) For $z \ge 0$, it corresponds to $P[|X| < z]$ where $X \sim \mathcal{N}(0, \frac{1}{2})$.

(c) $\lim\limits_{z \to \infty} \operatorname{erf}(z) = 1$

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z	$Q(z)$	z	$Q(z)$	z	$Q(z)$	z	$Q(z)$	z	$Q(z)$
3.00	$1.35 \cdot 10^{-3}$	3.40	$3.37 \cdot 10^{-4}$	3.80	$7.23 \cdot 10^{-5}$	4.20	$1.33 \cdot 10^{-5}$	4.60	$2.11 \cdot 10^{-6}$
3.01	$1.31 \cdot 10^{-3}$	3.41	$3.25 \cdot 10^{-4}$	3.81	$6.95 \cdot 10^{-5}$	4.21	$1.28 \cdot 10^{-5}$	4.61	$2.01 \cdot 10^{-6}$
3.02	$1.26 \cdot 10^{-3}$	3.42	$3.13 \cdot 10^{-4}$	3.82	$6.67 \cdot 10^{-5}$	4.22	$1.22 \cdot 10^{-5}$	4.62	$1.92 \cdot 10^{-6}$
3.03	$1.22 \cdot 10^{-3}$	3.43	$3.02 \cdot 10^{-4}$	3.83	$6.41 \cdot 10^{-5}$	4.23	$1.17 \cdot 10^{-5}$	4.63	$1.83 \cdot 10^{-6}$
3.04	$1.18 \cdot 10^{-3}$	3.44	$2.91 \cdot 10^{-4}$	3.84	$6.15 \cdot 10^{-5}$	4.24	$1.12 \cdot 10^{-5}$	4.64	$1.74 \cdot 10^{-6}$
3.05	$1.14 \cdot 10^{-3}$	3.45	$2.80 \cdot 10^{-4}$	3.85	$5.91 \cdot 10^{-5}$	4.25	$1.07 \cdot 10^{-5}$	4.65	$1.66 \cdot 10^{-6}$
3.06	$1.11 \cdot 10^{-3}$	3.46	$2.70 \cdot 10^{-4}$	3.86	$5.67 \cdot 10^{-5}$	4.26	$1.02 \cdot 10^{-5}$	4.66	$1.58 \cdot 10^{-6}$
3.07	$1.07 \cdot 10^{-3}$	3.47	$2.60 \cdot 10^{-4}$	3.87	$5.44 \cdot 10^{-5}$	4.27	$9.77 \cdot 10^{-6}$	4.67	$1.51 \cdot 10^{-6}$
3.08	$1.04 \cdot 10^{-3}$	3.48	$2.51 \cdot 10^{-4}$	3.88	$5.22 \cdot 10^{-5}$	4.28	$9.34 \cdot 10^{-6}$	4.68	$1.43 \cdot 10^{-6}$
3.09	$1.00 \cdot 10^{-3}$	3.49	$2.42 \cdot 10^{-4}$	3.89	$5.01 \cdot 10^{-5}$	4.29	$8.93 \cdot 10^{-6}$	4.69	$1.37 \cdot 10^{-6}$
3.10	$9.68 \cdot 10^{-4}$	3.50	$2.33 \cdot 10^{-4}$	3.90	$4.81 \cdot 10^{-5}$	4.30	$8.54 \cdot 10^{-6}$	4.70	$1.30 \cdot 10^{-6}$
3.11	$9.35 \cdot 10^{-4}$	3.51	$2.24 \cdot 10^{-4}$	3.91	$4.61 \cdot 10^{-5}$	4.31	$8.16 \cdot 10^{-6}$	4.71	$1.24 \cdot 10^{-6}$
3.12	$9.04 \cdot 10^{-4}$	3.52	$2.16 \cdot 10^{-4}$	3.92	$4.43 \cdot 10^{-5}$	4.32	$7.80 \cdot 10^{-6}$	4.72	$1.18 \cdot 10^{-6}$
3.13	$8.74 \cdot 10^{-4}$	3.53	$2.08 \cdot 10^{-4}$	3.93	$4.25 \cdot 10^{-5}$	4.33	$7.46 \cdot 10^{-6}$	4.73	$1.12 \cdot 10^{-6}$
3.14	$8.45 \cdot 10^{-4}$	3.54	$2.00 \cdot 10^{-4}$	3.94	$4.07 \cdot 10^{-5}$	4.34	$7.12 \cdot 10^{-6}$	4.74	$1.07 \cdot 10^{-6}$
3.15	$8.16 \cdot 10^{-4}$	3.55	$1.93 \cdot 10^{-4}$	3.95	$3.91 \cdot 10^{-5}$	4.35	$6.81 \cdot 10^{-6}$	4.75	$1.02 \cdot 10^{-6}$
3.16	$7.89 \cdot 10^{-4}$	3.56	$1.85 \cdot 10^{-4}$	3.96	$3.75 \cdot 10^{-5}$	4.36	$6.50 \cdot 10^{-6}$	4.76	$9.68 \cdot 10^{-7}$
3.17	$7.62 \cdot 10^{-4}$	3.57	$1.78 \cdot 10^{-4}$	3.97	$3.59 \cdot 10^{-5}$	4.37	$6.21 \cdot 10^{-6}$	4.77	$9.21 \cdot 10^{-7}$
3.18	$7.36 \cdot 10^{-4}$	3.58	$1.72 \cdot 10^{-4}$	3.98	$3.45 \cdot 10^{-5}$	4.38	$5.93 \cdot 10^{-6}$	4.78	$8.76 \cdot 10^{-7}$
3.19	$7.11 \cdot 10^{-4}$	3.59	$1.65 \cdot 10^{-4}$	3.99	$3.30 \cdot 10^{-5}$	4.39	$5.67 \cdot 10^{-6}$	4.79	$8.34 \cdot 10^{-7}$
3.20	$6.87 \cdot 10^{-4}$	3.60	$1.59 \cdot 10^{-4}$	4.00	$3.17 \cdot 10^{-5}$	4.40	$5.41 \cdot 10^{-6}$	4.80	$7.93 \cdot 10^{-7}$
3.21	$6.64 \cdot 10^{-4}$	3.61	$1.53 \cdot 10^{-4}$	4.01	$3.04 \cdot 10^{-5}$	4.41	$5.17 \cdot 10^{-6}$	4.81	$7.55 \cdot 10^{-7}$
3.22	$6.41 \cdot 10^{-4}$	3.62	$1.47 \cdot 10^{-4}$	4.02	$2.91 \cdot 10^{-5}$	4.42	$4.94 \cdot 10^{-6}$	4.82	$7.18 \cdot 10^{-7}$
3.23	$6.19 \cdot 10^{-4}$	3.63	$1.42 \cdot 10^{-4}$	4.03	$2.79 \cdot 10^{-5}$	4.43	$4.71 \cdot 10^{-6}$	4.83	$6.83 \cdot 10^{-7}$
3.24	$5.98 \cdot 10^{-4}$	3.64	$1.36 \cdot 10^{-4}$	4.04	$2.67 \cdot 10^{-5}$	4.44	$4.50 \cdot 10^{-6}$	4.84	$6.49 \cdot 10^{-7}$
3.25	$5.77 \cdot 10^{-4}$	3.65	$1.31 \cdot 10^{-4}$	4.05	$2.56 \cdot 10^{-5}$	4.45	$4.29 \cdot 10^{-6}$	4.85	$6.17 \cdot 10^{-7}$
3.26	$5.57 \cdot 10^{-4}$	3.66	$1.26 \cdot 10^{-4}$	4.06	$2.45 \cdot 10^{-5}$	4.46	$4.10 \cdot 10^{-6}$	4.86	$5.87 \cdot 10^{-7}$
3.27	$5.38 \cdot 10^{-4}$	3.67	$1.21 \cdot 10^{-4}$	4.07	$2.35 \cdot 10^{-5}$	4.47	$3.91 \cdot 10^{-6}$	4.87	$5.58 \cdot 10^{-7}$
3.28	$5.19 \cdot 10^{-4}$	3.68	$1.17 \cdot 10^{-4}$	4.08	$2.25 \cdot 10^{-5}$	4.48	$3.73 \cdot 10^{-6}$	4.88	$5.30 \cdot 10^{-7}$
3.29	$5.01 \cdot 10^{-4}$	3.69	$1.12 \cdot 10^{-4}$	4.09	$2.16 \cdot 10^{-5}$	4.49	$3.56 \cdot 10^{-6}$	4.89	$5.04 \cdot 10^{-7}$
3.30	$4.83 \cdot 10^{-4}$	3.70	$1.08 \cdot 10^{-4}$	4.10	$2.07 \cdot 10^{-5}$	4.50	$3.40 \cdot 10^{-6}$	4.90	$4.79 \cdot 10^{-7}$
3.31	$4.66 \cdot 10^{-4}$	3.71	$1.04 \cdot 10^{-4}$	4.11	$1.98 \cdot 10^{-5}$	4.51	$3.24 \cdot 10^{-6}$	4.91	$4.55 \cdot 10^{-7}$
3.32	$4.50 \cdot 10^{-4}$	3.72	$9.96 \cdot 10^{-5}$	4.12	$1.89 \cdot 10^{-5}$	4.52	$3.09 \cdot 10^{-6}$	4.92	$4.33 \cdot 10^{-7}$
3.33	$4.34 \cdot 10^{-4}$	3.73	$9.57 \cdot 10^{-5}$	4.13	$1.81 \cdot 10^{-5}$	4.53	$2.95 \cdot 10^{-6}$	4.93	$4.11 \cdot 10^{-7}$
3.34	$4.19 \cdot 10^{-4}$	3.74	$9.20 \cdot 10^{-5}$	4.14	$1.74 \cdot 10^{-5}$	4.54	$2.81 \cdot 10^{-6}$	4.94	$3.91 \cdot 10^{-7}$
3.35	$4.04 \cdot 10^{-4}$	3.75	$8.84 \cdot 10^{-5}$	4.15	$1.66 \cdot 10^{-5}$	4.55	$2.68 \cdot 10^{-6}$	4.95	$3.71 \cdot 10^{-7}$
3.36	$3.90 \cdot 10^{-4}$	3.76	$8.50 \cdot 10^{-5}$	4.16	$1.59 \cdot 10^{-5}$	4.56	$2.56 \cdot 10^{-6}$	4.96	$3.52 \cdot 10^{-7}$
3.37	$3.76 \cdot 10^{-4}$	3.77	$8.16 \cdot 10^{-5}$	4.17	$1.52 \cdot 10^{-5}$	4.57	$2.44 \cdot 10^{-6}$	4.97	$3.35 \cdot 10^{-7}$
3.38	$3.62 \cdot 10^{-4}$	3.78	$7.84 \cdot 10^{-5}$	4.18	$1.46 \cdot 10^{-5}$	4.58	$2.32 \cdot 10^{-6}$	4.98	$3.18 \cdot 10^{-7}$
3.39	$3.49 \cdot 10^{-4}$	3.79	$7.53 \cdot 10^{-5}$	4.19	$1.39 \cdot 10^{-5}$	4.59	$2.22 \cdot 10^{-6}$	4.99	$3.02 \cdot 10^{-7}$

Table 3.2 The standard normal complementary CDF $Q(z)$.

(d) $\operatorname{erf}(-z) = -\operatorname{erf}(z)$

(e) $\Phi(x) = \frac{1}{2}\left(1 + \operatorname{erf}\left(\frac{x}{\sqrt{(2)}}\right)\right) = \frac{1}{2}\operatorname{erfc}\left(-\frac{x}{\sqrt{2}}\right)$

(f) The complementary error function:
$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = 2Q\left(\sqrt{2}z\right) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}dx$
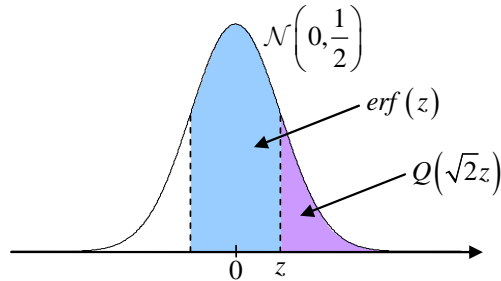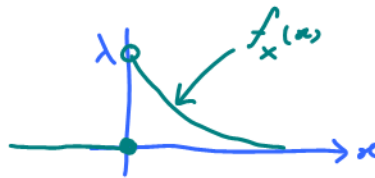

Figure 23: erf-function and Q-function

10.4.3 Exponential Distribution

Definition 10.53. The exponential distribution is denoted by $\mathcal{E}(\lambda)$.

(a) $\lambda > 0$ is a parameter of the distribution, often called the **rate parameter**.

(b) Characterized by

- $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$

- $F_X(x) = \begin{cases} 1 - e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0 \end{cases}$

$F_X(x) \equiv P[X \leq x] = \int_{-\infty}^{x} f_X(t)dt = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$

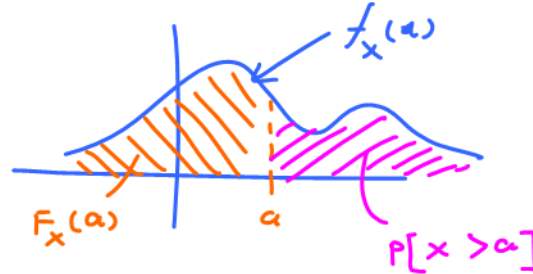$\int e^{-\lambda t}dt = \frac{1}{-\lambda}e^{-\lambda t}$

when $x \geq 0$

$F_X(x) = \int_{-\infty}^{x} f_X(t)dt = \int_0^x \lambda e^{-\lambda t}dt = \lambda\left(-\frac{1}{\lambda}\right)e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}$

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- Survival-, survivor-, or reliability-function:

$$ccdf \equiv 1 - cdf = 1 - F_X(x) = 1 - P[X \le x]$$

complementary

$$= P[X > x]$$

For $X \sim \mathcal{E}(\lambda)$

$$ccdf = \begin{cases} 1, & x < 0, \\ e^{-\lambda x} & x \ge 0. \end{cases}$$

$f_X(a)$

$F_X(a)$ a $P[x > a]$

(c) MATLAB:

- $X = $ `exprnd`$(1/\lambda)$ or `random('exp',`$1/\lambda)$
- $f_X(x) = $ `exppdf(x,`$1/\lambda)$ or `pdf('exp',x,`$1/\lambda)$
- $F_X(x) = $ `expcdf(x,`$1/\lambda)$ or `cdf('exp',x,`$1/\lambda)$

Example 10.54. Suppose $X \sim \mathcal{E}(\lambda)$, find $P[1 < X < 2]$.

Method ① : $P[1 < x < 2] = \int_1^2 \lambda e^{-\lambda x} dx = \left. \dfrac{\lambda e^{-\lambda x}}{-\lambda} \right|_1^2 = e^{-\lambda} - e^{-2\lambda}$

Method ② : $P[1 < x < 2] = F_X(2) - F_X(1) = \left(1 - e^{-2\lambda}\right) - \left(1 - e^{-\lambda}\right)$

Exercise 10.55. Exponential random variable as a continuous version of geometric random variable: Suppose $X \sim \mathcal{E}(\lambda)$. Show that $\lfloor X \rfloor \sim \mathcal{G}_0(e^{-\lambda})$ and $\lceil X \rceil \sim \mathcal{G}_1(e^{-\lambda})$
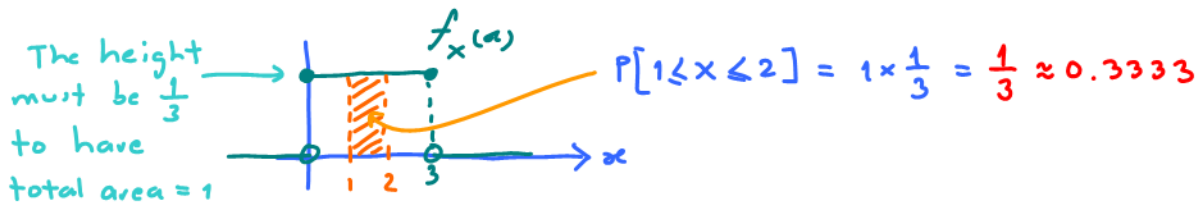
Example 10.56. The exponential distribution is intimately related to the Poisson process. It is often used as a probability model for the (waiting) time until a "rare" event occurs.

- time elapsed until the next earthquake in a certain region

- decay time of a radioactive particle

- time between independent events such as arrivals at a service facility or arrivals of customers in a shop.

- duration of a cell-phone call

- time it takes a computer network to transmit a message from one node to another.

Evaluate $P[1 < X < 2]$ for each of the following RVs

(a) $X \sim \mathcal{U}(0,3)$

The height must be $\frac{1}{3}$ to have total area = 1

$f_X(x)$

$P[1 \leq x \leq 2] = 1 \times \frac{1}{3} = \frac{1}{3} \approx 0.3333$

(b) $X \sim \mathcal{E}(3)$ $\underset{\lambda}{\curvearrowleft}$

$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 3e^{-3x}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$

$P[1 \leq x \leq 2] = \int_1^2 f_X(x)\,dx = \int_1^2 3e^{-3x}\,dx = \left. 3\frac{e^{-3x}}{-3} \right|_1^2 = e^{-3} - e^{-6} \approx 0.0473$

(c) $X \sim \mathcal{N}(0,1)$

$P[1 \leq x \leq 2] = \Phi(2) - \Phi(1) \approx 0.97725 - 0.8413 \approx 0.1359$

(d) $X \sim \mathcal{N}(1,3)$

$P[1 \leq x \leq 2] = \Phi\left(\frac{2-1}{\sqrt{3}}\right) - \Phi\left(\frac{1-1}{\sqrt{3}}\right) = \Phi\left(\frac{1}{\sqrt{3}}\right) - \Phi(0) \approx \Phi(0.58) - 0.5$

$\approx 0.7190 - 0.5 = 0.2190$

10.57. $\mathbb{E}X = \frac{1}{\lambda}$

look at Ex. 10.29

Example 10.58. Phone Company A charges \$0.15 per minute for telephone calls. For any fraction of a minute at the end of a call, they charge for a full minute. Phone Company B also charges \$0.15 per minute. However, Phone Company B calculates its charge based on the exact duration of a call. If T, the duration of a call in minutes, is exponential with parameter $\lambda = 1/3$, what are the expected revenues per call $\mathbb{E}[R_A]$ and $\mathbb{E}[R_B]$ for companies A and B?

Solution: First, note that $\mathbb{E}T = \frac{1}{\lambda} = 3$. Hence,

$$\mathbb{E}[R_B] = \mathbb{E}[0.15 \times T] = 0.15\mathbb{E}T = \$0.45.$$

and

$$\mathbb{E}[R_A] = \mathbb{E}[0.15 \times \lceil T \rceil] = 0.15\mathbb{E}\lceil T \rceil.$$

Now, recall, from Exercise 10.55, that $\lceil T \rceil \sim \mathcal{G}_1\left(e^{-\lambda}\right)$. Hence, $\mathbb{E}\lceil T \rceil = \frac{1}{1-e^{-\lambda}} \approx 3.53$. Therefore,

$$\mathbb{E}[R_A] = 0.15\mathbb{E}\lceil T \rceil \approx 0.5292.$$

10.59. *Memoryless property*: The exponential r.v. is the only continuous[44] r.v. on $[0, \infty)$ that satisfies the memoryless property:

$$P[X > s + x \,|\, X > s] = P[X > x]$$

for all $x > 0$ and all $s > 0$ [18, p. 157–159]. In words, the future is independent of the past. The fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen.

- Imagining that the exponentially distributed random variable X represents the lifetime of an item, the residual life of an item has the same exponential distribution as the original lifetime,

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[44]For discrete random variable, geometric random variables satisfy the memoryless property.

regardless of how long the item has been already in use. In other words, there is no deterioration/degradation over time. If it is still currently working after 20 years of use, then today, its condition is "just like new".

- In particular, suppose we define the set $B+x$ to be $\{x + b : b \in B\}$. For any $x > 0$ and set $B \subset [0, \infty)$, we have

$$P[X \in B + x | X > x] = P[X \in B]$$

because

$$\frac{P[X \in B + x]}{P[X > x]} = \frac{\int_{B+x} \lambda e^{-\lambda t} dt}{e^{-\lambda x}} \overset{\tau=t-x}{=} \frac{\int_B \lambda e^{-\lambda(\tau+x)} d\tau}{e^{-\lambda x}}.$$

10.60. Summary:

$X \sim$	Support S_X	$f_X(x) =$
Uniform $\mathcal{U}(a,b)$	(a,b)	$\begin{cases} \frac{1}{b-a}, & a < x < b, \\ 0, & \text{otherwise.} \end{cases}$
Normal (Gaussian) $\mathcal{N}(m,\sigma^2)$	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$
Exponential $\mathcal{E}(\lambda)$	$(0,\infty)$	$\begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \le 0 \end{cases}$

Table 4: Examples of probability density functions. Here, $\lambda, \sigma > 0$.